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3-Manifolds fibering over the Klein bottle and codimension 2 orientable fibrators

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Abstract

Let N be an orientable circle-bundle over the Klein bottle with obstruction b. The aim of this paper is to show that N is a codimension 2 orientable fibrator if and only if b is nonzero even.

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1. Introduction

A map $p:M \to B$ between locally compact ANRs is said to have an *approximate homotopy lifting property* (AHLP) *with respect to the space* X if, for each an open cover ε of B, and two maps $h:X \to M$ and $H:X \times [0,1] \to B$ such that $H_0 = p \circ h$, there exists a map $\widetilde{H}:X \times [0,1] \to M$ such that $\widetilde{H}_0 = h$ and that $p \circ \widetilde{H}$ is ε-close to H. If $p:M \to B$ is a proper map and has the AHLP with respect to all spaces, we say that p is an *approximate fibration*. By [4], it is known that if $p:M \to B$ is an approximate fibration and B is path-connected, then all fibers are shape equivalent.

From [6], we have the following: Let $p:M \to B$ be a proper map on an $(n+k)$-manifold such that each $p^{-1}(x)$ has the shape of some closed n-manifold N. Then for each $k \leqslant 2$, B is a k-manifold with (possibly empty) boundary. When $k \leqslant 2$, it is natural to ask the following:

Question. When is a proper map $p:M \to B$ an approximate fibration?

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Daverman introduced the following definition. A closed connected n-manifold N is called a *codimension k fibrator* (*codimension k orientable fibrator*, respectively) if each proper map $p : M \to B$ on an (orientable, respectively) $(n+k)$-manifold M onto a finite dimensional space B each fiber of which is shape equivalent to N is an approximate fibration. In the last decade, several articles have been devoted to the study of codimension 2 fibrators.

In [7, Example 6.1], Daverman proved that not every orientable S^1-bundle over the torus T is a codimension 2 orientable fibrator. By [6, Theorem 4.2], some twisted S^1-bundle over the Klein bottle K are not codimension 2 fibrators.

Let N be an orientable S^1-bundle over K with fiber F and let $p_N : N \to K$ be the projection map. Let E be a disk in K. Set $K' = \mathrm{Cl}(K \setminus E)$ and $N' = p_N^{-1}(K')$. We see that $p_N^{-1}(E)$ is homeomorphic to $E \times S^1$ and that N' is homeomorphic to a twisted S^1-bundle $K' \widetilde{\times} S^1$ over K'. We have a section $c : K' \to K' \widetilde{\times} S^1$. We see that $\partial(K' \widetilde{\times} S^1)$ is homeomorphic to $\partial c(K') \times S^1$. And we have a homeomorphism $f : \partial E \times S^1 \to \partial c(K') \times S^1$ such that N is homeomorphic to $E \times S^1 \cup_f K' \widetilde{\times} S^1$. Fix three orientations on $\partial c(K')$, S^1 and ∂E which are induced by one on N. Choose $x_1 \in \partial E$, $x_2 \in \partial c(K')$, and $y_1, y_2 \in S^1$. And denote the homotopy classes $\ell_1 = [x_1 \times S^1]$, $m_1 = [\partial E \times y_1]$ in $\pi_1(\partial E \times S^1)$ and $\ell_2 = [x_2 \times S^1]$, $m_2 = [\partial c(K') \times y_2]$ in $\pi_1(\partial c(K') \times S^1)$. Moreover $f_\# : \pi_1(\partial E \times S^1) \to \pi_1(\partial c(K') \times S^1)$ induces a 2×2 matrix of the following form:

$$\begin{pmatrix} f_\#(m_1) \\ f_\#(\ell_1) \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & \varepsilon \end{pmatrix} \cdot \begin{pmatrix} m_2 \\ \ell_2 \end{pmatrix}.$$

Where $\varepsilon = \pm 1$ and b is an integer. The integer b is called the *obstruction class* of N. The main aim of this paper is to show the following result.

Theorem 1.1. *Let N be an orientable S^1-bundle over the Klein bottle with obstruction b. Then N is a codimension* 2 *orientable fibrator if and only if b is nonzero even.*

By [15, Theorem 5.3], every orientable S^1-bundle over the torus T or the Klein bottle K is a closed 3-manifold with Nil structure. Theorem 1.1 shows that Theorem 6.4 of [7] has errors. By [6, Theorem 4.2], it is known that every closed orientable manifold which cyclically covers itself can not be a codimension 2 orientable fibrator. In fact, we will prove that every orientable S^1-bundle over the Klein bottle K with odd obstruction cyclically covers itself (nontrivially). In the similar way, we have the following.

Proposition 1.2. *Let N be an S^1-bundle over the torus T with obstruction b. If p^2 divides b for some prime p, then there is a p^2-*1 *cyclic covering from N to N.*

It is a mistake to think that no S^1-bundle over the torus T regularly, cyclically covers itself. See [7, Corollary 6.3].

2. Preliminaries

Throughout this paper, all spaces are locally compact, separable metrizable, and all manifolds are finite-dimensional, connected and boundaryless. Whenever we allow boundary, the object will be called a manifold with boundary. E^k and S^k denote the k-dimensional Euclidean space and the k-sphere with the standard topology, respectively. And T denotes the torus and K denotes the Klein bottle. Homology is computed with integer coefficients. $\check{H}_k(X)$ and $\check{H}^k(X)$ mean the Čech homology and cohomology of a space X, respectively. $\check{\pi}_k(X)$ denotes the kth shape group of a space X.

For the sake of convenience, a proper map $p: M \to B$ is called a *codimension k map* if M is an $(n+k)$-manifold, B is finite dimensional and each $p^{-1}(x)$ has the shape of some closed n-manifold N.

Let N be a closed n-manifold. A proper map $p: M \to B$ is said to be an *N-like map* if each $p^{-1}(x)$ has the shape of N.

Let N and N' be closed orientable n-manifolds and let $f: N \to N'$ be a map. The (absolute) *degree* $\deg f$ of f is $|H_n(f)(1)| \geqslant 0$, where f induces a homomorphism $H_n(f): H_n(N) \cong \mathbb{Z} \to H_n(N') \cong \mathbb{Z}$. In general case, f induces a homomorphism $H_n(f : \mathbb{Z}_2): H_n(N : \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_n(N' : \mathbb{Z}_2) \cong \mathbb{Z}_2$. The *degree* mod 2 of f is defined by $H_n(f : \mathbb{Z}_2)(1)$. It is known that every degree one map from N to N' induces a π_1-epimorphism from $\pi_1(N)$ to $\pi_1(N')$ and that every degree one mod 2 map from N to N' induces an epimorphism from $H_1(N : \mathbb{Z}_2)$ to $H_1(N' : \mathbb{Z}_2)$.

A closed orientable n-manifold N is called *hopfian* if every degree one map $\psi: N \to N$ is a homotopy equivalence. It is known that every closed orientable manifold with finite fundamental group is hopfian [8, Theorem 2.2].

Let $p: M^{n+k} \to B$ be a codimension k map. For each $x \in B$ there exist a neighborhood U_x of x in B and a shape retraction $R_x: p^{-1}(U_x) \to p^{-1}(x)$, because $p^{-1}(x)$ is an FANR. We define the *continuity set C_p* of the map $p: M \to B$:

$$C_p = \big\{x \in B:\ \text{there exist a neighborhood } U_x \text{ of } x \text{ in } B \text{ and a shape retraction } R_x: p^{-1}(U_x) \to p^{-1}(x) \text{ such that } R_x|p^{-1}(x'): p^{-1}(x') \to p^{-1}(x) \text{ is a degree one map for all } x' \in U_x\big\}.$$

Theorem 2.1 [6, Proposition 2.8]. *Let $p: M \to B$ be a codimension 2 map on an orientable manifold such that each $p^{-1}(x)$ has the shape type of some closed orientable n-manifold N. Then the space B is a 2-manifold and $D_p = B \setminus C_p$ is locally finite in B.*

3. Orientable S^1-bundles over the Klein bottle or the torus

Let N be an orientable S^1-bundle over K with fiber F and obstruction b and let $\mathrm{incl}: F \to N$ be the inclusion map. See the first section for obstruction and some notations. We calculate the first homology group $H_1(N)$ of N. By the Seifert–van Kampen Theorem, we have the following presentation of $\pi_1(K' \mathbin{\widetilde{\times}} S^1)$:

$$\pi_1\big(K' \mathbin{\widetilde{\times}} S^1\big) \cong \big\langle a_1, b_1, a_2, b_2, c \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = c^b, \\ a_i c a_i^{-1} = c = b_i c b_i^{-1}\ (i = 1, 2)\big\rangle.$$

We see that $H_1(K' \mathbin{\widetilde{\times}} S^1) \cong \mathbb{Z}^2 \times \mathbb{Z}_2$. We may view $H_1(\partial E \times S^1)$ as $\langle \ell_1 \mid - \rangle \times \langle m_1 \mid - \rangle$ and $H_1(K' \mathbin{\widetilde{\times}} S^1) \times H_1(E \times S^1)$ as $\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}$. We have the Mayer–Vietoris exact sequence

$$\cdots \to H_1\big(\partial E \times S^1\big) \xrightarrow{j} H_1\big(K' \mathbin{\widetilde{\times}} S^1\big) \times H_1\big(E \times S^1\big) \to H_1(N) \to 0,$$

where $j(\ell_1) = (0, 0, 1, -1)$ and $j(m_1) = (\pm 2, \pm 2, |b|, 0)$. If b is even, then $H_1(N) \cong \mathbb{Z} \times \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}/\langle(1, -1)\rangle) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We notice that $\mathrm{incl}_*(H_1(F)) = 0 \times 0 \times \mathbb{Z}_2$. If b is odd, then $H_1(N) \cong \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}_2/\langle(2, 1)\rangle) \cong \mathbb{Z} \times \mathbb{Z}_4$. We notice that $\mathrm{incl}_*(H_1(F)) = 0 \times 2\mathbb{Z}_4$.

Let $\theta : N^* \to N$ be a k-1 covering. The covering θ induces the decomposition $G_\theta = \{C \mid C$ is a component of $(p_N \circ \theta)^{-1}(y)$ for $y \in K\}$ of N^* and the decomposition map $p_{N^*} : N^* \to K^* = N^*/G_\theta$. And we have the covering $\theta^* : K^* \to K$ satisfying the following commutative diagram:

```tikzcd
N^* \arrow[r, "\theta"] \arrow[d, "p_{N^*}"'] & N \arrow[d, "p_N"] \\
K^* \arrow[r, "\theta^*"'] & K
```

We see that N^* is an orientable S^1-bundle over K^* with the projection p_{N^*}. If $\theta^{-1}(F)$ has k' components, then k' divides k and θ^* is a k'-1 covering. By [15, Lemma 3.5], we have the following.

Lemma 3.1. *Let Y be a closed surface, let N be an orientable S^1-bundle over Y with fiber F and with obstruction b, and let $\theta : N^* \to N$ be a k-1 covering. And let k' be the number of components of $\theta^{-1}(F)$. Then N^* is an orientable S^1-bundle over a closed surface Y^* with obstruction $bk'^2/k = bk'/k''$ satisfying $\chi(Y^*) = k'\chi(Y)$, where $k'' = k/k'$ and $\chi(Y)$ is the Euler number of Y.*

Lemma 3.2. *Let $N(1)$ be an orientable S^1-bundle over K with odd obstruction and let $N(2)$ be an orientable S^1-bundle over K with even obstruction. Then there is no degree one map either from $N(1)$ to $N(2)$ or from $N(2)$ to $N(1)$.*

Proof. Since degree one maps induce surjections on homology with integer coefficients, this result follows from the information in Section 3. □

Lemma 3.3. *Let $N(i)$ be an orientable S^1-bundle over K with obstruction b_i for $i = 1, 2$. If $0 \leqslant b_1 < b_2$, then there is no degree one map from $N(1)$ to $N(2)$.*

Proof. We suppose that $f : N(1) \to N(2)$ is a degree one map. By taking the pullback of the 2-1 cover $T \to K$, we obtain 2-1 covering $\theta_2 : N(2)^* \to N(2)$ such that $N(2)^*$ is an orientable S^1-bundle over T with obstruction $2b_2$. There are a 2-1 covering $\theta_1 : N(1)^* \to$

$N(1)$ and a map $f^*: N(1)^* \to N(2)^*$ such that $\theta_2 \circ f^* = f \circ \theta_1$. We see from Lemma 3.1 that $H_1(N(1)^*)$ is isomorphic to either $\mathbb{Z} \times \mathbb{Z}_4$, $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}^2 \times \mathbb{Z}_{2b_1}$. Since f^* has degree one, $f^*_*: H_1(N(1)^*) \to H_1(N(2)^*)$ is an epimorphism. This is a contradiction because $H_1(N(2)^*) \cong \mathbb{Z}^2 \times \mathbb{Z}_{2b_2}$. □

Lemma 3.4. *Let $N(i)$ be an orientable S^1-bundle over K with obstruction $t_i 2^{r_i}$, where t_i is odd for $i = 1, 2$. If $r_1 \neq r_2$, then there is no degree one map from $N(1)$ to $N(2)$.*

Proof. We suppose that $f: N(1) \to N(2)$ is a degree one map. And by Lemma 3.2, we suppose that r_1 and r_2 are nonzero.

We see that $H_1(N(1)) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. There exists a 2-1 covering $\theta_1^{(1)}: N(1)^{(1)} \to N(1)$ such that $\operatorname{Im} \theta_1^{(1)}{}_{\#} = h^{-1}(\mathbb{Z} \times \mathbb{Z}_2 \times 0)$, where $h: \pi_1(N(1)) \to H_1(N(1))$ is the Hurewicz epimorphism. We see from Lemma 3.1 that $N(1)^{(1)}$ is an orientable S^1-bundle over K with obstruction $t_1 2^{r_1 - 1}$. Since $f_*: H_1(N(1)) \to H_1(N(2))$ is an isomorphism, there exist a 2-1 covering $\theta_2^{(1)}: N(2)^{(1)} \to N(2)$ and a degree one map $f^{(1)}: N(1)^{(1)} \to N(2)^{(1)}$ such that $\theta_2^{(1)} \circ f^{(1)} = f \circ \theta_1^{(1)}$. Since $f^{(1)}$ is a degree one map, $N(2)^{(1)}$ is an orientable S^1-bundle over K with obstruction $t_2 2^{r_2 - 1}$ or $t_2 2^{r_2 + 1}$.

We assume that $r_2 > r_1$. This shows that there exist two 2^{r_1}-1 coverings $\theta_1': N(1)' \to N(1)$ and $\theta_2': N(2)' \to N(2)$, and a degree one map $f': N(1)' \to N(2)'$ such that $\theta_2' \circ f' = f \circ \theta_1'$, that $N(1)'$ is an orientable S^1-bundle over K with obstruction t_1 and that $N(2)'$ is an orientable S^1-bundle over K with obstruction $t_2 2^{r'}$, where $r' > 0$. By Lemma 3.2, this is a contradiction.

We suppose that $r_1 > r_2$. There exists a 2-1 covering $\theta_2^*: N(2)^* \to N(2)$ such that $N(2)^*$ is an orientable S^1-bundle over K with obstruction $t_2 2^{r_2 - 1}$. Since $f^*: H_1(N(1)) \to H_1(N(2))$ is an isomorphism, there is a 2-1 covering $\theta_1^*: N(1)^* \to N(1)$ and a degree one map $f^*: N(1)^* \to N(2)^*$ such that $\theta_2^* \circ f^* = f \circ \theta_1^*$. Since $f_*: H_1(N(1)) \to H_1(N(2))$ is an isomorphism, by Lemma 3.1, $N(1)^*$ is an orientable S^1-bundle over K with obstruction either $t_1 2^{r_1 - 1}$ or $t_1 2^{r_1 + 1}$. This shows that there exist two 2^{r_2}-1 covering $\theta_2''': N(2)''' \to N(2)$, $\theta_1''': N(1)''' \to N(1)$ and a degree one map $f''': N(1)''' \to N(2)'''$ such that $\theta_2''' \circ f''' = f \circ \theta_1'''$, that $N(2)'''$ is an orientable S^1-bundle over K with obstruction t_2 and that $N(1)'''$ is an orientable S^1-bundle over K with obstruction $t_1 2^r$, where $r > 0$. By Lemma 3.2, this is a contradiction. □

Lemma 3.5. *Let N be an orientable S^1-bundle over the Klein bottle K with obstruction $t 2^r$, where t is odd. And let $\theta: N^* \to N$ be a k-1 covering such that N^* is an orientable S^1-bundle over K with obstruction $t' 2^r$, where t' is odd. If $r \geqslant 1$ and k is even, then θ is not cyclic.*

Proof. We think of K as an S^1-bundle over S^1. As a few lines before Lemma 3.1, we have a k'-1 covering $\theta^*: K \to K$ and an s'-1 covering $\theta^{**}: S^1 \to S^1$ satisfying the following commutative diagram:

$$
\begin{tikzcd}
N^* \arrow[r, "\theta"] \arrow[d, "p_{N^*}"'] & N \arrow[d, "p_N"] \\
K^* = K \arrow[r, "\theta^*"] \arrow[d, "p_{K^*}"'] & K \arrow[d, "p_K"] \\
S^1 \arrow[r, "\theta^{**}"] & S^1
\end{tikzcd}
$$

Let F be a fiber of p_N, let F' be a fiber of p_2 and let $k'' = k/k'$. Since N^* is an orientable S^1-bundle over K with obstruction $t'2^r$ and k is even, by Lemma 3.1, k' and k'' are even. And since θ^* is a covering from K to K, s' is odd. We see that $s'' = k'/s'$ is even. Let $\theta^{*-1}(F')_C$ be a component of $\theta^{*-1}(F')$. Since $\theta^*|\theta^{*-1}(F')_C : \theta^{*-1}(F')_C \to F'$ is an s''-1 covering, $0 \times \mathbb{Z}_2 \subset \operatorname{Ker}\theta^*_* \subset H_1(K) \cong \mathbb{Z} \times \mathbb{Z}_2$. Thus there exists an odd integer $m \geqslant 3$ such that $\operatorname{Im}\theta^*_* = \langle (m, \varepsilon)\rangle$, where $\varepsilon = 0, 1$ and $(m, \varepsilon) \in H_1(K) \cong \mathbb{Z} \times \mathbb{Z}_2$. Let $\theta^{-1}(F)_C$ be a component of $\theta^{-1}(F)$. Since k'' is even and $\theta|\theta^{-1}(F)_C : \theta^{-1}(F)_C \to F$ is a k''-1 covering, $0 \times 0 \times \mathbb{Z}_2 \subset \operatorname{Ker}\theta_* \subset H_1(N^*) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, $\operatorname{Im}\theta_* = \theta_*(\mathbb{Z} \times \mathbb{Z}_2 \times 0) = \langle (m, \varepsilon, \varepsilon')\rangle$, where $\varepsilon' = 0, 1$ and $(m, \varepsilon, \varepsilon') \in H_1(N) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\varepsilon = 0$ or $\varepsilon' = 0$, θ is not cyclic. We assume that $\varepsilon = \varepsilon' = 1$. Since there is an isomorphism $\xi : \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying that $\xi(m, 1, 1) = (m, 1, 0)$, θ is not cyclic. □

Lemma 3.6. *Every nontrivial orientable S^1-bundle over K is hopfian.*

Proof. Let N be a nontrivial orientable S^1-bundle over K. We see from [10, Theorem 15.19] that $\pi_1(N)$ is residually finite, thus, it is hopfian. Since N is aspherical, N is hopfian. □

Next let N be an orientable S^1-bundle over T with fiber F and let $p_N : N \to T$ be the projection map. Let E be a disk in T. Set $T' = \mathrm{Cl}(T \setminus E)$ and $N' = p_N^{-1}(T')$. We see that N' and $p_N^{-1}(E)$ are homeomorphic to $T' \times S^1$ and $E \times S^1$, respectively. And we have a homeomorphism $f : \partial E \times S^1 \to \partial T' \times S^1$ such that N is homeomorphic to $E \times S^1 \cup_f T' \times S^1$. Fix three orientations on $\partial T'$, S^1 and ∂E which are induced by one on N. Choose $x_1 \in \partial E$, $x_2 \in \partial T'$, and $y_1, y_2 \in S^1$. And denote the homotopy classes $\ell_1 = [x_1 \times S^1]$, $m_1 = [\partial E \times y_1]$ in $\pi_1(\partial E \times S^1)$ and $\ell_2 = [x_2 \times S^1]$, $m_2 = [\partial T' \times y_2]$ in $\pi_1(\partial T' \times S^1)$.

Moreover $f_\# : \pi_1(\partial E \times S^1) \to \pi_1(\partial T' \times S^1)$ induces a 2×2 matrix of the following form:

$$\begin{pmatrix} f(m_1) \\ f(\ell_1) \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} m_2 \\ \ell_2 \end{pmatrix}.$$

Where $\varepsilon = \pm 1$ and b is an integer. The integer b is called the *obstruction class* of N. Since N is nontrivial, $b \neq 0$. By the Seifert–van Kampen Theorem, we have the following presentation of $\pi_1(N)$:

$$\pi_1(N) \cong \bigl\langle a_1, b_1, a_2, b_2, c \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = c^b, \quad a_i c a_i^{-1} = c = b_i c b_i^{-1}\ (i = 1, 2)\bigr\rangle. \tag{$*$}$$

Since F is connected, $(p_N)_*: H_1(N) \to H_1(T)$ is an epimorphism. Because $H_1(T) \cong \mathbb{Z}^2$, $(p_N)_*: H_1(N) \to H_1(T)$ is split. We have that $H_1(N) \cong H_1(T) \times \operatorname{Ker}(p_N)_*$. It follows from $(*)$ that $H_1(F_g)$ is generated by $\{h(a_1), h(b_1), \ldots, h(a_g), h(b_g)\}$, and that $\operatorname{Ker}(p_N)_*$ is generated by $h(c)$, where $h: \pi_1(N) \to H_1(N)$ is the Hurewicz homomorphism. We see that $H_1(N) \cong H_1(T) \times \mathbb{Z}/|b|\mathbb{Z} \cong \mathbb{Z}_2 \times Z_{|b|}$ and notice that $\operatorname{incl}_*(H_1(F)) = 0 \times 0 \times \mathbb{Z}_{|b|}$, where $\operatorname{incl}: F \to N$ is the inclusion map.

4. Orientable S^1-bundles over the Klein bottle and codimension 2 orientable fibrators

Lemma 4.1. *Let N be an orientable S^1-bundle over the Klein bottle K with even obstruction $b \geqslant 1$. Let M be an orientable 5-manifold and let $p: M \to E^2$ be an N-like map such that $p|M \setminus p^{-1}(0): M \setminus p^{-1}(0) \to E^2 \setminus 0$ is an approximate fibration. Then $p: M \to E^2$ is an approximate fibration.*

Proof. Since $p|M \setminus p^{-1}(0): M \setminus p^{-1}(0) \to E^2 \setminus 0$ is an approximate fibration, we have a shape strong deformation retraction $R: M \to p^{-1}(0)$ which is a shape equivalence. Fix an element $x \in E^2 \setminus 0$ and set $g_0 = p^{-1}(0)$ and $g = p^{-1}(x)$. We think of g_0 and g as N. Let $p_N: N \to K$ be the projection map. By Lemma 3.6, N is hopfian. From [5, Theorem 3.9], it suffices to show that $R|g: g \to g_0$ has degree one.

We show that $R|g: g \to g_0$ has positive degree. From [8, Lemma 5.2′], it suffices to show that $A = \operatorname{Im}\{(p_N \circ R|g)_*: \check{H}_1(g) \to \check{H}_1(g_0) \to H_1(K)\}$ is infinite. We suppose that A is finite. We have a covering $q_1: M_1 \to M$ satisfying $\operatorname{Im} q_{1\#} \cong ((p_N \circ R)_* \circ h)^{-1}(A)$, where $h: \pi_1(M) \to H_1(M)$ is the Hurewicz epimorphism. Denote $g_0^1 = q_1^{-1}(g_0)$. Since R is a shape equivalence, as the proof of [3, Lemma 3.5], there exists a covering $\theta_1: N(1) \to N$ such that $\operatorname{Im} q_{1\#} \cong \operatorname{Im} \theta_{1\#}$ and $\check{H}_C^k(g_0^1) \cong H_C^k(N(1))$ for each $k \geqslant 1$. The covering θ_1 induces the decomposition $G_{\theta_1} = \{C \mid C \text{ is a component of } (p_N \circ \theta_1)^{-1}(y) \text{ for } y \in K\}$ of $N(1)$ and the decomposition map $p_{N(1)}: N(1) \to K_1 = N(1)/G_{\theta_1}$. We have a covering $\theta_1^*: K_1 \to K$ satisfying the following commutative diagram:

$$\begin{tikzcd} N(1) \arrow[r, "\theta_1"] \arrow[d, "p_{N(1)}"'] & N \arrow[d, "p_N"] \\ K_1 \arrow[r, "\theta_1^*"'] & K \end{tikzcd}$$

We notice that $N(1)$ is an orientable S^1-bundle over K_1. Let $h': \pi_1(K) \to H_1(K)$ be the Hurewicz epimorphism. Since θ_1 is infinite and $\operatorname{Im} {\theta_1}^*{}_\# \cong h'^{-1}(A)$, θ_1^* is infinite and K_1 is homeomorphic to $S^1 \times (0,1)$. This shows that $N(1)$ is homeomorphic to $S^1 \times S^1 \times (0,1)$. And the covering q_1 induces the decomposition

$$G_{q_1} = \big\{(p \circ q_1)^{-1}(x')_C \mid (p \circ q_1)^{-1}(x')_C \\ \text{is a component of } (p \circ q_1)^{-1}(x') \text{ for } x' \in E^2\big\}$$

of M_1 and the decomposition map $p_1 : M_1 \to B_1 = M_1/G_{q_1}$. We have a natural commutative diagram:

$$\begin{tikzcd} M_1 \setminus g_0^1 \arrow[r, "q_1"] \arrow[d, "p_1|"'] & M \setminus g_0 \arrow[d, "p|"] \\ B_1 \setminus p_1(g_0^1) \arrow[r, "q_1^*"'] & E^2 \setminus 0 \end{tikzcd}$$

Since q_1^* is an infinite covering, $B_1 \setminus p_1(g_0^1)$ is homeomorphic to an open disk. Since $p_1|M_1 \setminus g_0^1 : M_1 \setminus g_0^1 \to B_1 \setminus p_1(g_0^1)$ is an N-like approximate fibration, $M_1 \setminus g_0^1$ has the homotopy type of N. By duality (see [14, p. 296]),

$$H_1\big(M_1, M_1 \setminus g_0^1\big) \cong \check{H}_C^4\big(g_0^1\big) \cong H_C^4\big(N(1)\big) = 0.$$

We notice that M_1 has the homotopy type of $N(1)$. From the homology exact sequence of $(M_1, M_1 \setminus g_0^1)$, we have an exact sequence

$$H_1\big(M_1 \setminus g_0^1\big) \cong H_1(N) \to H_1(M_1) \cong H_1\big(N(1)\big) \to H_1\big(M_1, M_1 \setminus g_0^1\big) = 0.$$

Since $N(1)$ is homeomorphic to $S^1 \times S^1 \times (0,1)$, we have an exact sequence $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z} \times \mathbb{Z} \to 0$. This is a contradiction. We conclude that $R|g : g \to g_0$ has positive degree.

Since $R|g : g \to g_0$ has positive degree, $k_2 = \#\check{\pi}_1(g_0)/\operatorname{Im}\{(R|g)_\# : \check{\pi}_1(g) \to \check{\pi}_1(g_0)\}$ is finite. We have a k_2-1 covering $q_2 : M_2 \to M$ satisfying $\operatorname{Im} q_{2\#} \cong \operatorname{Im}(R|g)_\#$, and a shape strong deformation retraction $R_2 : M_2 \to g_0^2 = q_2^{-1}(g_0)$ which is a shape equivalence and satisfies $q_2 \circ R_2 = R \circ q_2$. And there exists a k_2-1 covering $\theta_2 : N(2) \to N$ such that g_0^2 has the shape of $N(2)$. We notice that every component g_C^2 of $g^2 = q_2^{-1}(g)$ has the shape of N, and that the number of the components of g^2 is k_2. Fix a component g_C^2 of $g^2 = q_2^{-1}(g)$. Since $q_2|g_C^2 : g_C^2 \to g$ is a homeomorphism, $(R_2|g_C^2)_\# : \check{\pi}_1(g_C^2) \to \check{\pi}_1(g_0^2)$ is an epimorphism. Because $(R_2|g_C^2)_* : \check{H}_1(g_C^2) \to \check{H}_1(g_0^2)$ is an epimorphism, $N(2)$ is an orientable S^1-bundle over K with even obstruction b'. And the covering q_2 induces the decomposition $G_{q_2} = \{(p \circ q_2)^{-1}(x')_C \mid (p \circ q_2)^{-1}(x')_C$ is a component of $(p \circ q_2)^{-1}(x')$ for $x' \in E^2\}$ of M_2 and the decomposition map $p_2 : M_2 \to B_2 = M_2/G_{q_2}$. Since $(R_2|g_C^2)_* : \check{H}_1(g_C^2) \to \check{H}_1(g_0^2)$ is an isomorphism, by [1, Lemma 3.4], $R_2|g_C^2 : g_C^2 \to g_0^2$ has degree one. We conclude that $R|g : g \to g_0$ has degree k_2.

Next we suppose that k_2 is even. Let $b = t2^r$, where t is odd and $r \geqslant 1$. From Lemma 3.4, we see that $N(2)$ is an orientable S^1-bundle over K with obstruction $t'2^r$, where t' is odd. By Lemma 3.5, we have that θ_2 is not cyclic. From [4, Corollary 3.5], we have an exact sequence

$$\pi_2\big(E^2 \setminus 0\big) \cong 1 \to \check{\pi}_1(g) \xrightarrow{i_\#} \pi_1(M \setminus g_0) \to \pi_1\big(E^2 \setminus 0\big) \cong \mathbb{Z} \to 1.$$

This show that $i_\#(\check{\pi}_1(g))$ is a normal subgroup of $\pi_1(M \setminus g_0)$. It follows from the proof of [8, Lemma 5.1] or [3, Proposition 3.5] that the homomorphism $\pi_1(M \setminus g_0) \to \pi_1(M)$ induced by the inclusion map is an epimorphism. Since $R_\#\pi_1(M) \to \check{\pi}_1(g_0)$ is an isomorphism, $\operatorname{Im}(R|g)_\#$ is a normal subgroup of $\check{\pi}_1(g_0)$ and we have a commutative diagram:

$$\begin{array}{ccccccc}
\check{\pi}_1(g) & \xrightarrow{i_\#} & \pi_1(M \setminus g_0) & \longrightarrow & \pi_1(M) & \xrightarrow{R_\#} & \check{\pi}_1(g_0) \\
 & & \downarrow & & & & \downarrow \\
 & \mathbb{Z} \cong & \pi_1(M \setminus g_0)/i_\#(\check{\pi}_1(g)) & & \longrightarrow & & \check{\pi}_1(g_0)/\operatorname{Im}(R|g)_\#
\end{array}$$

This shows that $\check{\pi}_1(g_0)/\operatorname{Im}(R|g)_\#$ is cyclic. We have a contradiction because θ_2 is not cyclic.

We suppose that $k_2 \geqslant 3$ is odd. Denote $A' = \operatorname{Im}\{(R|g)_* : \check{H}_1(g) \to \check{H}_1(g_0)\}$. Because $\check{H}_1(R|g;\mathbb{Z}_2)_* : \check{H}_1(g;\mathbb{Z}_2) \to \check{H}_1(g_0;\mathbb{Z}_2)$ is an epimorphism, there exists a positive integer $k_1 \geqslant 3$ which is odd such that $A' = (k_1\mathbb{Z}) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \check{H}_1(g_0)$. We notice that $A' = (p_N)_*^{-1}(A)$ and that q_1 is a k_1-1 covering. We have a shape strong deformation retraction $R_1 : M_1 \to g_0^1 = q_1^{-1}(g_0)$ which is a shape equivalence and satisfies $q_1 \circ R_1 = R \circ q_1$. We see that $\theta_1 : N(1) \to N$ is a k_1-1 covering such that g_0^1 has the shape of $N(1)$. It follows from Lemma 3.1 that $N(1)$ is an orientable S^1-bundle over K with obstruction bk_1. We notice that every component g_C^1 of $g^1 = q_1^{-1}(g)$ has the shape type of N. And we see that $(q_1|g_0^1)_* : \check{H}_1(g_0^1) \to A' \subset \check{H}_1(g_0)$ is an isomorphism. Fix a component g_C^1 of g^1. Since $q_1|g_C^1 : g_C^1 \to g$ is a homeomorphism, $(R_1|g_C^1)_* : \check{H}_1(g_C^1) \to \check{H}_1(g_0^1)$ is an isomorphism. Since $(R_1|g_C^1)_* : \check{H}_1(g_C^1) \to \check{H}_1(g_0^1)$ is an isomorphism, by [1, Lemma 3.4], $R_1|g_C^1 : g_C^1 \to g_0^1$ has degree one. By Lemma 3.3, this is a contradiction. We conclude that $R|g : g \to g_0$ has degree one. $\square$

Proof of Theorem 1.1. Let N be an orientable S^1-bundle over K with even obstruction $b \geqslant 2$ and let $p : M \to B$ be an N-like map on an orientable 5-manifold. By Lemma 3.6, N is hopfian. And from [8, Theorem 2.1], we see that $p : M \to B$ is an approximate fibration over C_p. It follows from Theorem 2.1 and Lemma 4.1 that p is an approximate fibration, that is, N is a codimension 2 orientable fibrator.

Next let N be an orientable S^1-bundle over K with odd obstruction b, let $\theta : N(1) \to N$ be a 4-1 cyclic covering satisfying $\operatorname{Im}\theta_\# \cong ((p_N)_* \circ h)^{-1}(\mathbb{Z} \times 0)$, where $h : \pi_1(N) \to H_1(N)$ is the Hurewicz epimorphism and $H_1(N) \cong \mathbb{Z} \times \mathbb{Z}_4$. We have a finite covering $\theta^* : K \to K$ with a commutative diagram:

$$\begin{array}{ccc}
N(1) & \xrightarrow{\theta} & N \\
{\scriptstyle p_{N(1)}}\downarrow & & \downarrow{\scriptstyle p_N} \\
K & \xrightarrow[\theta^*]{} & K
\end{array}$$

We note that θ^* is a 2-1 or 4-1 covering. Let F be a fiber of p_N and let $\mathrm{incl}_* : H_1(F) \to H_1(N)$ be the homomorphism induced by the inclusion map $\mathrm{incl} : F \to N$. Since $\operatorname{Im}\mathrm{incl}_*$ is finite, θ^* is a 2-1 covering. Thus, we see from Lemma 3.1 that $N(1)$ is an orientable S^1-bundle over K with obstruction b. By [6, Theorem 4.2], N is not a codimension 2 orientable fibrator.

Finally, let N be an orientable S^1-bundle over K with zero obstruction. Since N cyclically covers itself, by [6, Theorem 4.2], N is not a codimension 2 orientable fibrator. $\square$

Proof of Proposition 1.2. Let N be an S^1-bundle over T with positive obstruction b such that p^2 divides b for some prime p. We see that $H_1(N) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_b$. There exist epimorphisms $d_1 : \mathbb{Z} \to \mathbb{Z}_{p^2}$ and $d_3 : \mathbb{Z}_b \to \mathbb{Z}_p$ and the monomorphism $k : \mathbb{Z}_p \to \mathbb{Z}_{p^2}$. Let $t : \mathbb{Z} \to \mathbb{Z}_{p^2}$ be the trivial map and let $r = d_1 \times t \times (k \circ d_3) : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_b \to \mathbb{Z}_{p^2}$ be an epimorphism. Let $\theta : N(1) \to N$ be a p^2-1 cyclic covering satisfying $\operatorname{Im}\theta_\# \cong \operatorname{Ker}(r \circ h)$, where $h : \pi_1(N) \to H_1(N)$ is the Hurewicz epimorphism. We have a finite covering $\theta* : T \to T$ with a diagram:

$$\begin{tikzcd}
N(1) \arrow[r, "\theta"] \arrow[d, "p_{N(1)}"'] & N \arrow[d, "p_N"] \\
T \arrow[r, "\theta^*"'] & T
\end{tikzcd}$$

Let F be a fiber of p_N and let $\mathrm{incl}_* : H_1(F) \to H_1(N)$ be the homomorphism induced by the inclusion map $\mathrm{incl} : F \to N$. And $\theta^{-1}(F)_C$ be a component of $\theta^{-1}(F)$. Since $\operatorname{Im}\mathrm{incl}_* = 0 \times 0 \times \mathbb{Z}_b \subset H_1(N)$, $\theta \mid \theta^{-1}(F)_C : \theta^{-1}(F)_C \to F$ is a p-1 covering. Thus, we see from Lemma 3.1 that $N(1)$ is an orientable S^1-bundle over T with obstruction b. $\square$

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